Multiplicative Equivariant Thom Spectra & Structured Real Orientations

joint with Ryan Quinn

§1 Motivation

麻辣烫

Problem 1.1 (May, 1974). Does BP admit an $\mathbb{E}_\infty$-algebra structure?

Brown–Peterson, 1966: Definition of BP.
May, 1974: Posted "Problems in infinite loop space theory".
Kriz, 1990s: Proposed strategy to obtain $\mathbb{E}_\infty$-structure via TAQ.
Hu–May–Kriz, 2001: MU is not $\mathbb{E}_\infty$-BP-algebra.
Basterra–Mandell, 2010: BP is $\mathbb{E}_4$.
Chadwick–Mandell, 2013: BP is $\mathbb{E}_2$ via Quillen idempotent.
Lawson (p=2), Senger (p odd), 2017: BP is not $\mathbb{E}_{2(p^2+2)}$.
Carmeli–Luecke, 2025: Quillen idempotent is not $\mathbb{E}_5$ at $p=3$.

HHR $\rightsquigarrow$ $MU_{\mathbb{R}}$, $BP_{\mathbb{R}}$

Problem 1.2. For which C_2-representations V does $BP_{\mathbb{R}}$ admit an $\mathbb{E}_V$-algebra structure?

Theorem 1.3 (Quinn–Z.). $BP_{\mathbb{R}}$ admits an $\mathbb{E}_\rho$-algebra structure.

Very rough strategy: $BP_{\mathbb{R}}$ is split off via the Real Quillen idempotent $MU_{\mathbb{R}} \longrightarrow MU_{\mathbb{R}}$. Show that it is $\mathbb{E}_\rho$.

multiplicative equivariant Thom spectra

structured orientations

§2 Review: Multiplicative Thom Spectra

Def 2.1 (Ando–Blumberg–Gepner–Hopkins–Rezk, 2008).

R ring spectrum

$f: X \to \mathrm{Pic}\, R$ map of spaces

$$\mathrm{Th}(f) = Mf = \mathrm{colim}\left(X \xrightarrow{f} \mathrm{Pic}\, R \longrightarrow \mathrm{LMod}_R\right)$$

in Cat_∞

important: Grothendieck's homotopy hypothesis

§2.1 Universal Property of Multiplicative Thom Spectra

Ando–Blumberg–Gepner 2011, Antolín-Camarena—Barthel 2014: Multiplicative version

Def 2.2. Let $R \in \mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathrm{Sp})$.

(i) $\mathrm{Pic}\, R = GL_1(\mathrm{LMod}_R^{\mathrm{core}})$ Picard space $\rightsquigarrow$ It is $\mathbb{E}_n$

(ii) $A \in \mathrm{LMod}_R$

$$\begin{tikzcd} \mathrm{Pic}(R)_{\downarrow A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{LMod}_{R/A} \arrow[d] \\ \mathrm{Pic}(R) \arrow[r] & \mathrm{LMod}_R \end{tikzcd}$$

Objects: $M \to A$ of R-modules, M invertible

Theorem 2.3 (Antolín-Camarena—Barthel, 2014).

$R \in \mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathrm{Sp})$

X $\mathbb{E}_n$-space $\quad f: X \to \mathrm{Pic}\, R$ map of $\mathbb{E}_n$-spaces

(i) $\mathrm{Th}(f)$ inherits $\mathbb{E}_n$-structure.

(ii) $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{LMod}_R) \Rightarrow \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{LMod}_R)}(\mathrm{Th}(f), A) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})/\mathrm{Pic}\, R}(X, \mathrm{Pic}(R)_{\downarrow A})$

$\mathrm{Th}(f) \to A \quad \rightsquigarrow$

$$\begin{tikzcd} & \mathrm{Pic}(R)_{\downarrow A} \arrow[d] \\ X \arrow[ur, dashed, "\mathbb{E}_n"] \arrow[r, "f"'] & \mathrm{Pic}(R) \end{tikzcd}$$

Proof Sketch. Operadic left Kan extensions!

Theorem 2.4 (Lurie).

$\mathcal{O}^\otimes$ ∞-operad

$\mathcal{C}^\otimes, \mathcal{D}^\otimes$ $\mathcal{O}$-monoidal ∞-cats, $p: \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ structure map

$\mathcal{D}^\otimes$ $\mathcal{O}$-distributive

(i) $\mathrm{Fun}^{\mathrm{lax}}_{/\mathcal{O}^\otimes}(\mathcal{C}^\otimes, \mathcal{D}^\otimes) \underset{p^*}{\overset{p_!}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{lax}}_{/\mathcal{O}^\otimes}(\mathcal{O}^\otimes, \mathcal{D}^\otimes)$ ($p_! \dashv p^*$) operadic left Kan extension

$$\begin{tikzcd} \mathcal{C}^\otimes \arrow[r] \arrow[d, "p"'] & \mathcal{D}^\otimes \\ \mathcal{O}^\otimes \arrow[ur, dashed] & \end{tikzcd}$$

(ii) $\mathcal{O}^\otimes$ singly-colored $\Rightarrow$ $p_! F$ enhances $\mathrm{colim}(F|_{\mathcal{C}}: \mathcal{C} \to \mathcal{D})$

$F \in \mathrm{Fun}^{\mathrm{lax}}_{/\mathcal{O}^\otimes}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$

Operadically left Kan extend

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & \mathrm{Pic}\, R \arrow[r] & \mathrm{LMod}_R \\ \mathbb{E}_n^{\otimes} \arrow[rru, dashed, "p_! f = \mathrm{Th}(f)^{\otimes}"'] & & \end{tikzcd}$$

Operadic Lan & universal property of monoidal slice cats yields the theorem

□

§2.2 Abstract Orientation Theory

Def 2.5. $R \in \mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathrm{Sp})$

$R \to A$ map of $\mathbb{E}_{n+1}$-ring spectra

$X \to \mathrm{Pic}\, R$ map of $\mathbb{E}_n$-spaces

$\mathbb{E}_n$-orientation of A is one of the following equivalent characterizations

(i) Nullhomotopy of $X \to \mathrm{Pic}(R) \xrightarrow{\mathrm{Ind}_R^A} \mathrm{Pic}(A)$ of $\mathbb{E}_n$-spaces

(ii) $\mathbb{E}_n$-lift

$$\begin{tikzcd} & \mathrm{GL}_1(\mathrm{Pic}(R)_{/A}) \arrow[d] \\ X \arrow[ru, dashed] \arrow[r, "f"'] & \mathrm{Pic}(R) \end{tikzcd}$$

(iii) $\mathrm{Th}(f) \to A$ in $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{LMod}_R)$ s.t. for all $x: * \to X$ the adjoint A-module map corresponding to the R-module map

$$\mathrm{Th}(f \circ x) \longrightarrow \mathrm{Th}(f) \longrightarrow A$$

is an eq.

$\rightsquigarrow$ **$\mathrm{Or}_A^{\mathbb{E}_n}(f)$**

Consequences 2.6. Let $f: X \to \mathrm{Pic}\, R$ be an $\mathbb{E}_n$-map.

* X grouplike (connected if $n=0$) $\Longrightarrow$ $\mathrm{Or}_A^{\mathbb{E}_n}(f) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{LMod}_R)}(\mathrm{Th}(f), A)$
* $\mathbb{E}_n$-orientation of A of f $\rightsquigarrow$ $A \otimes_R \mathrm{Th}(f) \simeq A \otimes \Sigma^\infty_+ X$ in $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{LMod}_A)$

Thom isomorphism

Example 2.7.

* $\mathrm{id}_{MU}: MU \to MU$ $\quad$ $\mathbb{E}_\infty$-orientation
* $MU \otimes_{\mathbb{S}} MU \simeq MU \otimes_{\mathbb{S}} \Sigma^\infty_+ BU$ $\quad$ in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Sp})$

§3 Crash Course in Parametrized Higher Algebra

§3.1 Equivariant Higher Category Theory

Elmendorff $\mathcal{S}_G \simeq \mathrm{Fun}(\mathrm{Orb}_G^{\mathrm{op}}, \mathcal{S})$ motivates

$$BU_{\mathbb{R}}: \begin{array}{c} BU_{\mathbb{R}}(C_2/C_2) = BU_{\mathbb{R}}^{C_2} = BO \\ \downarrow \\ BU_{\mathbb{R}}(C_2/e) = BU_{\mathbb{R}}^{e} = BU \circlearrowleft C_2 \end{array}$$

Def 3.1 (Barwick-Dotto-Glasman-Nardin-Shah, 2016.)

(i) $\mathrm{Cat}_{G,\infty} = \mathrm{Fun}(\mathrm{Orb}_G^{\mathrm{op}}, \mathrm{Cat}_\infty)$ $\quad$ G-∞-categories

(ii) G-functor is a natural transformation

Example 3.2 (Equivariant Grothendieck Hypothesis).

G-space $\mathrm{Orb}_G^{\mathrm{op}} \to \mathcal{S}$ is G-∞-cat $\mathrm{Orb}_G^{\mathrm{op}} \to \mathcal{S} \to \mathrm{Cat}_\infty$

Example 3.3. $\mathcal{S}_G$, Sp_G

$G = C_2$: Sp_{C_2} is

$$\begin{array}{c} \mathrm{Sp}_{C_2}(C_2/C_2) = \mathrm{Sp}_{C_2} \\ \downarrow \\ \mathrm{Sp}_{C_2}(C_2/e) = \mathrm{Sp} \circlearrowleft C_2 \end{array}$$

Can import all usual notions of category theory, e.g. G-co-/limits.

Example 3.4. Let $H \leq G$. There is a G-∞-cat

$$\underline{*}_H(G/K) = \begin{cases} * & K \leq H, \\ \emptyset & K \not\leq H. \end{cases}$$

$\underline{*}_H \to \mathrm{Sp}_G \rightsquigarrow X \in \mathrm{Sp}_H$,

$\underline{\mathrm{colim}}(\underline{*}_H \to \mathrm{Sp}_G) \simeq \mathrm{Ind}_H^G X = \coprod_{G/H} X$

§3.2 Equivariant Higher Algebra

Def 3.5 (Nardin-Shah).

(i) $\mathbb{F}_{G,*}$ at $H \leq G$ is $\mathbb{F}_{H,*}$

(ii) G-∞-operad $:\Leftrightarrow$ G-∞-cat $\underline{\mathcal{O}}^{\otimes}$ with a G-functor $\underline{\mathcal{O}}^{\otimes} \to \underline{\mathbb{F}}_{G,*}$ satisfying certain operad conditions

Example 3.6.

(i) $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\infty} \rightsquigarrow \mathrm{Infl}_G \mathcal{O}^{\otimes} \in \mathrm{Op}_{G,\infty}$ an algebra over $\mathrm{Infl}_G \mathcal{O}^{\otimes}$ is a levelwise $\mathcal{O}$-algebra

(ii) N_{∞}-operad is G-∞-operad

$\mathbb{E}_{\infty}^{G}$ terminal G-∞-operad

(iii) V G-rep $\rightsquigarrow$ $\mathbb{E}_V \in \mathrm{Op}_{G,\infty}$

Relation between $+$ & $\cdot$

colim & $\otimes$

distributivity $\rightsquigarrow$ Nardin-Shah, Lenz-Linskens-Pützstück

Def 3.7. $(\mathcal{C}, \otimes)$ monoidal ∞-cat with colimits

distributive $:\Leftrightarrow$ for $F: I \to \mathcal{C}$, $G: J \to \mathcal{C}$ with colim diagrams $I^{\triangleright} \to \mathcal{C}$, $J^{\triangleright} \to \mathcal{C}$:

$$(I \times J)^{\triangleright} \longrightarrow I^{\triangleright} \times J^{\triangleright} \longrightarrow \mathcal{C} \times \mathcal{C} \xrightarrow{- \otimes -} \mathcal{C}$$

is colim diagram

Core point: $\operatorname{colim}_I F \otimes \operatorname{colim}_J G$

Restriction to $I \times J$: $F \otimes G$

$\Rightarrow$ Distributivity: $\operatorname{colim}_I F \otimes \operatorname{colim}_J G \simeq \operatorname{colim}_{I \times J} F \otimes G \simeq \operatorname{colim}_I \operatorname{colim}_J F \otimes G$,

i.e. $- \otimes -$ commutes with colims in each variable

Equivariantly: More delicate

Lemma 3.8.

$\underline{\mathcal{C}}^{\otimes}$ distributive C_2-sym mon C_2-∞-cat, $A \in \underline{\mathcal{C}}^{\otimes}_{C_2}$, $X \in \underline{\mathcal{C}}^{\otimes}_{e}$

$\Rightarrow$

* $\mathrm{Ind}_e^{C_2}(\mathrm{Res}_e^{C_2} A \otimes X) \simeq A \otimes \mathrm{Ind}_e^{C_2} X$
* $N_e^{C_2}(A \oplus B) \simeq N_e^{C_2} A \oplus \mathrm{Ind}_e^{C_2}(A \otimes B) \oplus N_e^{C_2} B$

$(a+b)^2 = a^2 + 2ab + b^2$

Theorem 3.9 (Nardin, 2017). Sp_G has a distributive G-sym mon structure.

§4 Equivariant Thom Spectra

Goal: Equivariantize ACB

Def 4.1.

Horev-Klang-Zou: $R = \mathbb{S}$

R G-ring spectrum

$f : X \to \mathrm{Pic}_G(R)$ map of G-spaces

$$\mathrm{Th}_G(f) = Mf = \mathrm{colim}\left(X \xrightarrow{f} \underline{\mathrm{Pic}_G(R)} \longrightarrow \underline{\mathrm{LMod}_R}\right)$$

§4.1 Equivariant Module Tech

$$\begin{tikzcd}
\mathrm{LMod}_R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Alg}_{\mathcal{LM}}(\mathrm{Sp}) \arrow[d] \\
* \arrow[r, "R"'] & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})
\end{tikzcd}$$

$\mathcal{LM}^{\otimes}, \mathbb{E}_1^{\otimes} \rightsquigarrow \mathrm{Infl}_G \mathcal{LM}^{\otimes}, \mathrm{Infl}_G \mathbb{E}_1^{\otimes}$

Observation 4.2.

$\mathcal{O}^{\otimes} \in \mathrm{Op}_\infty$

$R \in \mathrm{Alg}_{\mathcal{O} \otimes \mathbb{E}_1}(\mathrm{Sp}) \rightsquigarrow \mathcal{O}^{\otimes} \to \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})^{\otimes}$

$$\begin{tikzcd}
\mathrm{LMod}_R^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Alg}_{\mathcal{LM}}(\mathrm{Sp})^{\otimes} \arrow[d] \\
\mathcal{O}^{\otimes} \arrow[r, "R"'] & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})^{\otimes}
\end{tikzcd}
\quad \text{in } \mathrm{Op}_\infty \implies
\begin{tikzcd}
\mathrm{LMod}_R^{\otimes} \arrow[d] \\
\mathcal{O}^{\otimes}
\end{tikzcd}
\ \mathcal{O}^{\otimes}\text{-monoidal } \infty\text{-category}$$

Need:
$$\begin{tikzcd}
\mathrm{LMod}_R^{\otimes} \arrow[d] \\
\mathcal{O}^{\otimes}
\end{tikzcd}
\text{ coCart fib } \Longleftarrow
\begin{tikzcd}
\mathrm{Alg}_{\mathcal{LM}}(\mathrm{Sp})^{\otimes} \arrow[d] \\
\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})^{\otimes}
\end{tikzcd}
\text{ coCart fib}$$

Without $(-)^{\otimes}$: Lurie

With: Use Haugseng-Melani-Safronov criterion

Steward's work + Equivariantize!

Theorem 4.3. $\mathrm{LMod}_R^{\otimes}$ distributive

Proof Sketch. Consider colim diagrams $I^{\triangleright}, J^{\triangleright} \to \mathrm{LMod}_R$

$$\begin{tikzcd}
(I * J)^{\triangleright} \arrow[r] \arrow[d, equal] & I^{\triangleright} \times J^{\triangleright} \arrow[r] \arrow[d, equal] & \mathrm{LMod}_R^{\times 2} \arrow[r, "\otimes"] \arrow[d] & \mathrm{LMod}_{R \otimes R} \arrow[r, "R \otimes_{R\otimes R} -"] \arrow[d] & \mathrm{LMod}_R \\
(I * J)^{\triangleright} \arrow[r] & I^{\triangleright} \times J^{\triangleright} \arrow[r] & \mathrm{Sp}^{\times 2} \arrow[r, "\otimes"'] & \mathrm{Sp} &
\end{tikzcd}$$

Goal: Top composite is colim

Know: Bottom composite is colim by distributivity of Sp

$\xRightarrow[R \otimes_{R\otimes R} - \text{ preserves colims}]{\downarrow \text{ reflects colims}}$ Top composite is colim $\blacksquare$

Equivariantize!

§4.2 Multiplicative Equivariant Thom Spectra

Theorem 4.4 (Quinn–Z., 2025).

$R \in \mathrm{Alg}_{\mathbb{E}_{V+1}}(\mathrm{Sp})$

X $\mathbb{E}_V$-space

$f : X \to \underline{\mathrm{Pic}}_G(R)$ map of $\mathbb{E}_V$-spaces

(i) $\mathrm{Th}_G(f)$ inherits $\mathbb{E}_V$-structure.

(ii) $A \in \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathrm{LMod}}_R^G) \Rightarrow \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathrm{LMod}}_R^G)}(\mathrm{Th}_G(f), A) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_V}(\mathcal{S}_G)/\underline{\mathrm{Pic}}_G(R)}(X, \underline{\mathrm{Pic}}_G(R)_{\downarrow A})$

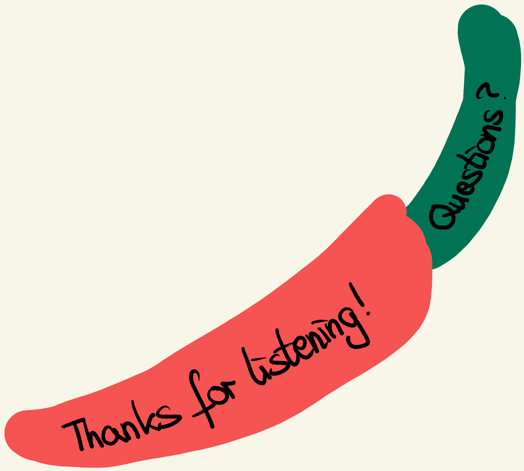
$$\begin{tikzcd}
 & \underline{\mathrm{Pic}}_G(R)_{\downarrow A} \arrow[d] \\
X \arrow[r, "f"'] \arrow[ur, dotted, "\mathbb{E}_V"] & \underline{\mathrm{Pic}}_G(R)
\end{tikzcd}$$

Consequences 4.5. Let $f : X \to \underline{\mathrm{Pic}}_G(R)$ be an $\mathbb{E}_V$-map.

* X grouplike (connected if $V = 0$) $\Rightarrow \mathrm{Or}_A^{\mathbb{E}_V}(f) \simeq \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathrm{LMod}}_R^G)}(\mathrm{Th}_G(f), A)$

  $\Rightarrow \mathrm{id} : \mathrm{Th}_G(f) \to \mathrm{Th}_G(f)$ is $\mathbb{E}_V$-orientation

* $\mathbb{E}_V$-orientation of A of $f \rightsquigarrow \underbrace{A \otimes_R \mathrm{Th}_G(f) \simeq A \otimes X}_{\text{Thom isomorphism}}$ in $\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathrm{LMod}}_A^G)$


Thanks for listening!
Questions?